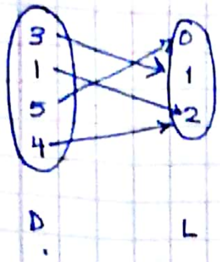
eg

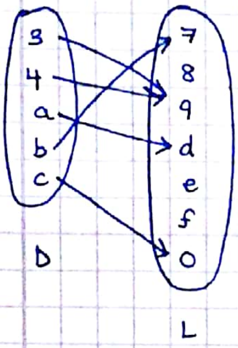


this is a Function.

∴ Functions can be:
- one – to – one
- many – to – one

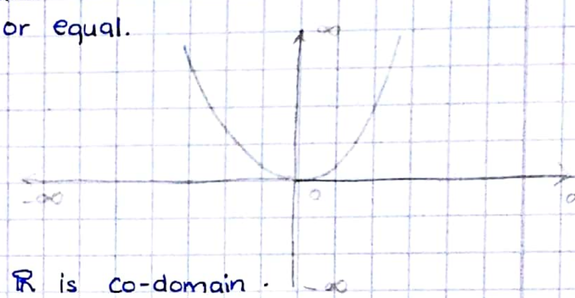
Functions cannot be:
- one-to-many

eg



is a function.
Domain = D
Co-domain = L
Range = $\{7, 9, d, 0\}$ = Image

Range $\subseteq$ co-domain.
↳ subset or equal.



Function representation: $f: \mathbb{R} \rightarrow \mathbb{R}$
$f(x) = x^2$

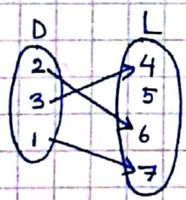
where the first $\mathbb{R}$ is the domain and second $\mathbb{R}$ is co-domain.
x-axis is domain and y axis is co-domain. Range from $0 \rightarrow \infty$.
Here, Range $\neq$ co-domain (no negative f value is an image from an element in domain)
Range $[0, \infty)$. Range $\subseteq$ co-domain.

Definition: Assume $f: D \rightarrow L$ is a function. We say ~~L~~ f is onto (surjective) if co-domain = Range.

eg: $f: \mathbb{R} \rightarrow [0, \infty)$
$f(x) = x^2$ is now onto. bc codomain = range

(injective)
We say f is 1-1 if:
- 2 differents elements in the domain correspond to 2 different elements in the co-domain.
- ~~For~~ each element in the range corresponds to one and only one element in the domain.
- whenever $f(a_1) = f(a_2)$, for some $a_1, a_2 \in$ Domain, then $a_1 = a_2$

eg:



this is a function.
function is one to one.
function is not onto.

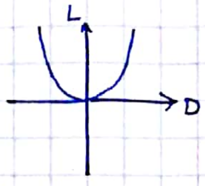
Domain = $\{2, 3, 1\}$
Co-domain = $\{4, 5, 6, 7\}$
Range = $\{4, 6, 7\}$

eg: $f: \mathbb{R} \rightarrow [0, \infty)$
$f(x) = x^2$ is onto
is not one to one

horizontal line check for 1-1

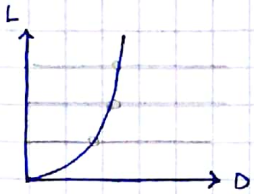
Def: $f: D \to L$ is called a <u>bijective</u> function if it is 1-1 AND onto

eg $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$.



- not onto Range $[0, \infty) \neq \mathbb{R}$
- not 1-1

eg: $f: [0,\infty) \to [0,\infty)$
$f(x) = x^2$



- is onto
- is one-to-one
- is <u>bijective</u>

Def: Assume $f: D \to L$ is a <u>bijective</u> function.
then $f^{-1}: L \to D$ is a function that is also a <u>bijective</u> function.

note: f^{-1} <u>does not</u> mean $\frac{1}{f}$

Suppose two functions:
$\boxed{1}$ $f_1 = x^2$
$f_2 = x+1$ $\}$ Assume $f: \mathbb{R} \to \mathbb{R}$

composition

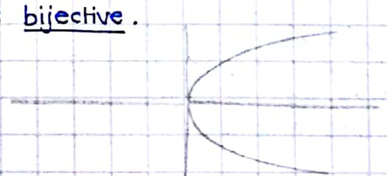
$f_1 \circ f_2 = f_1(f_2(x)) = f_1(x+1) = (x+1)^2 = x^2 + 2x + 1.$

$f_2 \circ f_1 = f_2(f_1(x)) = f_2(x^2) = x^2 + 1.$

Observe: the composition does not ~~compute~~ <u>commute</u> i.e $f_1 \circ f_2 \neq f_2 \circ f_1$
so <u>order matters</u>..

if $f_1: \mathbb{R} \to \mathbb{R}$
$f_1(x) = x^2$ does not have an inverse not bijective

but $f_1: [0,\infty) \to [0,\infty)$
$f_1(x) = x^2$ has an inverse bc f_1 is <u>bijective</u>.



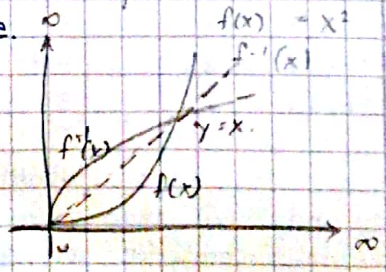
$f_1^{-1} = \underset{L}{[0,\infty)} \to \underset{D}{[0,\infty)}$

$y = f_1(x) = x^2$
swap: $x = y^2$
solve for y: $y = \pm\sqrt{x}$, we choose $+\sqrt{x}$ since Range $[0,\infty]$
$\therefore$ $f^{-1}(x) = \sqrt{x}$

now $f_1 \circ f_1^{-1} = f_1(\sqrt{x}) = x$
So, $f_1(x)$ and $f_1^{-1}(x)$ are symmetric <u>along the line $y = x$</u>

Note. $x^2 + y^2 = 4$ is <u>not</u> a function.
$\therefore$ cannot describe i~~s~~s as onto, 1-1 or bijective.

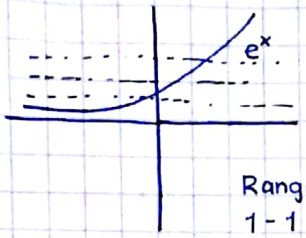
plot $f: [0,\infty) \to [0,\infty)$
$f(x) = x^2$
$f^{-1}(x)$



eg: $f: \mathbb{R} \to (0,\infty)$ 0 not included.
$f(x) = e^x$

is $f(x)$ bijective?
if yes, find $f^{-1}(x)$.

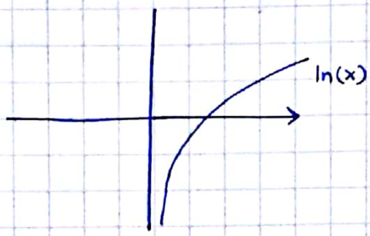
Ans: Draw.



Range $(0,\infty)$ = Codomain. ✓
1-1 ✓

yes $f(x)$ is bijective.

Finding $f^{-1}(x)$
$f: (0,\infty) \to \mathbb{R}$
$y = f(x) = e^x$
$\cancel{y}\ x = f^{-1}(y) = e^y$
solve for y: $y = \ln(x)$
$f^{-1}(x) = \ln(x)$



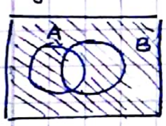
18-Mar-2018 Sets Boolean Algebra

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \longleftrightarrow$ • $S_1 \vee (S_2 \wedge S_3) \equiv (S_1 \vee S_2) \wedge (S_1 \vee S_3)$

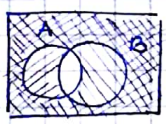
De Morgan

- $\overline{(A \cup B)} = \bar{A} \cap \bar{B} \longleftrightarrow$ • $(S_1 \vee S_2)' \equiv \cancel{S_1' A}\ S_1' \wedge S_2'$
- $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$ • $\neg(S_1 \wedge S_2) \equiv \neg S_1 \vee \neg S_2$

Venn diagram representation:



$\overline{(A \cap B)}$

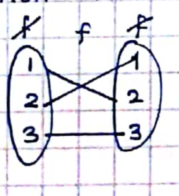


$\bar{A} \cup \bar{B}$

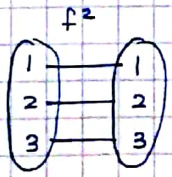
Prove set eqns by comparing elements in LHS and RHS

Prove boolean Algebra eqⁿ by Comparing truth tables

Suppose function f is 1-1 and onto (~~bijectio~~ bijective)



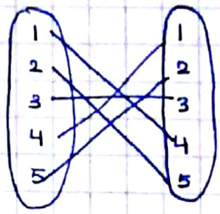
$f^2 = f \circ f$
then $(f \circ f)(1) = 1$
$(f \circ f)(2) = 2$
$(f \circ f)(3) = 3$



identity map $= I$ (i.e $y = x$)

Identity map will map each element to itself

Identity map composite any other bijective function will result in same bijective function eg: any ~~dt~~ bijective function with finite set can be represented as cycle



→ $f = (1\ 4)(2\ 5)$ (3 is not mentioned bc it maps to itself).

Ques: Find least integer value n s.t $f^n = I$

Ans: $n =$ ~~LCM(|set 1|, |set 2|)~~ LCM (| cycle 1 |, | cycle 2 |)

$= \text{LCM}(2, 2) = 2$

$\therefore\ f^2 = I$, $f \circ f = I$.

Eg: $f = (1\ 2\ 3)(4\ 6\ 5)(7\ 8)$

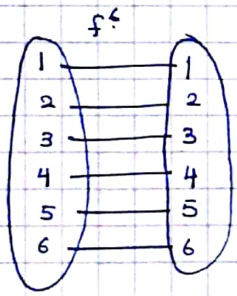
Domain:	1	2	3	4	5	6	7	8
Co-domain = Range:	2	3	1	6	~~4~~	~~4~~5	8	7

Find least integer value s.t $f^n = I$

$n = \text{LCM}(3, 3, 2) = 6$ i.e $f^6 = I$

check.

$f^2 =$	3	1	2	5	6	4	7	8
$f^3 =$	1	2	3	4	5	6	8	7
$f^4 =$	2	3	1	6	4	5	7	8
$f^5 =$	3	1	2	5	6	4	8	7
$f^6 =$	1	2	3	4	5	6	7	8



Identity Map

LCM [6, 8, 12, 14] :

$$\begin{array}{r|l} 2 & 6,\ 8,\ 12,\ 14 \\ 2 & 3,\ 4,\ 6,\ 7 \\ 2 & 3,\ 2,\ 3,\ 7 \\ 3 & 3,\ 1,\ 3,\ 7 \\ 7 & 1,\ 1,\ 1,\ 7 \\ & 1,\ 1,\ 1,\ 1 \end{array}$$

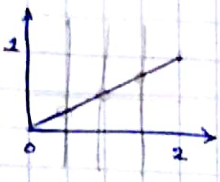
$\therefore$ LCM $= 2^3 \times 3 \times 7$

Homework :

1 (i) Let $f:(0,2) \to (0,1]$ s.t. $f(x) = 0{\cdot}5x$. If Is f a function? Is it 1 - 1 ? Is it onto ? Explain briefly.

ans :

vertical line check

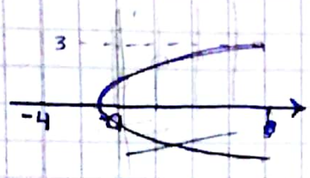


f is a function. It is 1 - 1 bc. every value in the co-domain is mapped back to exactly one element in the domain. f is not onto bc range $\neq$ co domain

The domain is almost 2 → image will be almost 1 but 1 is not mapped to anything in domain.

ii) let $f:(-4,8) \to (0,3)$ s.t $f(x) = \sqrt{x+1}$. is f a function? is it injective? is it Surjective? Explain.

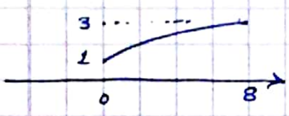
ans.



f is not a function bc for values $x < -1$, there is no corresponding value in the co domain ($x < -1$ has no image). ~~And the elements in the domain maps to more than one element in the codomain (one to many)~~

iii) Let $f:(0,8) \to (a\ b)$ s.t $f(x) = \sqrt{x+1}$. Find a, b so that f is bijective. Then find domain and range of f^{-1}. Write down the eq^n of f^{-1}.

ans



in order for f to be ~~bijectio~~ bijective $a = 1$ and $b = 3$.

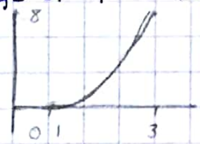
domain of $f^{-1} = (1, 3)$.

range of $f^{-1} = (0, 8)$.

$f^{-1} \Rightarrow \quad x = \sqrt{y+1}$

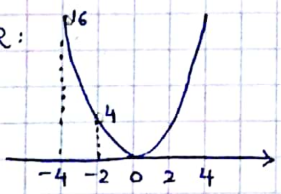
$x^2 - 1 = y$

$\therefore f^{-1}(x) = x^2 - 1$

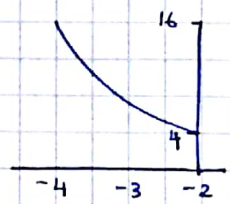


iv) Let $f:$ ~~R~~ $(-4, b) \to (a, 4)$ s.t $f(x) = x^2$. Find a, b so that f is bijective. Then find the domain and range of f^{-1}. Write down the eqn for f^{-1}.

ans : Assuming $\mathbb{R} \to \mathbb{R}$:



To make f bijective, $a = 16$, $b = -2$:



Domain of f^{-1} : $(16, 4)$

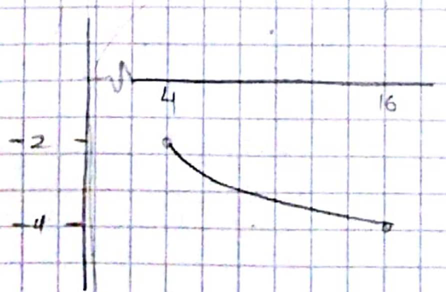
Range of f^{-1} : $(-4, -2)$

f^{-1} : $y = x^2$

$x = y^2$

$y = \mp\sqrt{x}$

$\therefore f^{-1}(x) = -\sqrt{x}$



v) Let $f : \{1\ 2\ 3\ 4\ 5\ 6\ 7\} \to \{1\ 2\ 3\ 4\ 5\ 6\ 7\}$ s.t
$f(1) = 2$, $f(2) = 1$, $f(3) = 4$, $f(4) = 5$, $f(5) = 6$, $f(6) = 7$ and $f(7) = 3$

i.e. $f == \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 4 & 5 & 6 & 7 & 3 \end{matrix}$

Find f^2 and f^3. Write f as a composition of disjoint cycles, then find the smallest possible integer $n \geq 1$ s.t. $f^n = I$

ans: F as disjoint ~~set~~ cycles: $(1\ 2)(3\ 4\ 5\ 6\ 7)$

$f^2 = 1 \quad 2 \quad 5 \quad 6 \quad 7 \quad 3 \quad 4$

$f^3 = 2 \quad 1 \quad 6 \quad 7 \quad 3 \quad 4 \quad 5$

$n = \text{LCM}(2, 5) = 10$ so $f^{10} = I$

20 - Mar - 2018 Def: Assume A and B are two sets.
We say $|A| = |B|$ iff there is a bijective function $f: A \to B$

We say A is countable if
- either A is a finite set
- or there exists a bijective function from $A \to N$ (or $A \to N^*$)

Otherwise A is uncountable:

Results: 1. Assume $|A| = \infty$ and $|B| = n < \infty$
then $|A \cup B| = |A|$

2. Assume $F_1, F_2, F_3, F_4, \ldots, F_n, F_{n+1}, F_{n+2}, \ldots$ are countable
Then $\cup F_i$'s is a countable set.

Statement: Z is countable
Proof: $f: Z \to N$

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ 2|n| - 1 & \text{if } n < 0 \end{cases}$$

Mapping +ve integers to even numbers and negative integers to odd numbers.

and $f(n)$ is bijective

can conclude $|Z| = |N|$
and Z is countable

Statement: Q is countable. Hence $|Q| = |Z| = |N|$
Proof: $F_1 = Z$, which is countable

$F_2 = \frac{1}{2} + Z = \{\frac{1}{2} + a \mid a \in Z\}$, countable

$F_3 = (\frac{1}{3} + Z) \cup (\frac{2}{3} + Z)$, countable

$F_4 = (\frac{1}{4} + Z) \cup (\frac{3}{4} + Z)$, countable

$F_5 = (\frac{1}{5} + Z) \cup (\frac{2}{5} + Z) \cup \cdots (\frac{4}{5} + Z)$, countable

$F_n = \cup (\frac{a}{n} + Z)$, $a < n$, $\gcd(a, n) = 1$, countable.

total # of elements with $\gcd(a, n) = 1$ is $\phi(n)$

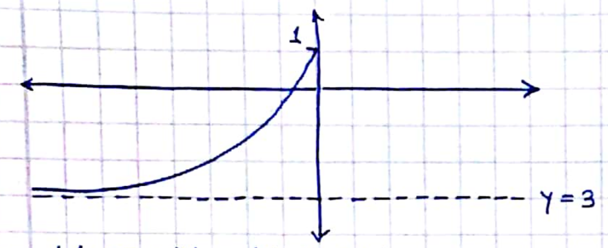
$\mathbb{Q} = \bigcup_{i \in \mathbb{N}^*} F_i$; countable

any rational number $\frac{a}{b}$, $\gcd(a,b) = 1$ can be found in F_b. Eg $-\frac{1}{2} = \frac{1}{2} + (-1) \in F_2$

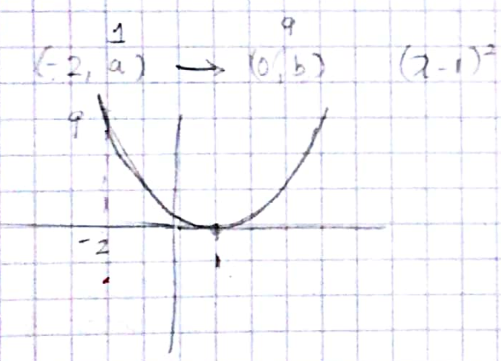
<u>Fact:</u> There is no bijective function f from $\mathbb{R} \to \mathbb{N}$
Hence $\mathbb{R}$ is uncountable and $|\mathbb{R}| \neq |\mathbb{N}|$

<u>Question:</u> Convince me that $|(-\infty, 0)| = |(-3, 1)| = |\mathbb{R}|$
we show $|(-\infty, 0)| = |(-3, 1]|$

Produce a function: $f: (-\infty, 0) \longrightarrow (-3, 1)$
$f(x) = 4e^x - 3$



Hence $|(-\infty, 0)| = |(-3, 1)|$
Side note: suppose asked to show $|(-\infty, 0]| = |(-3, 1)|$
Since $|(-\infty, 0)| = |(-3, 1)|$
and $(-\infty, 0) \cup \{0\} = (-\infty, 0]$
We know $|(-\infty, 0]| = |(-\infty, 0)| = |(-3, 1)|$



Homework 10

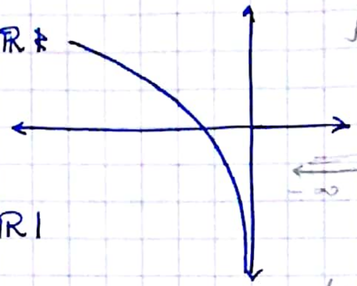
I i. Convince me that $|\mathbb{R}| = |(-4,1]|$

ans: first show ~~0, ∞~~ $|(-\infty,0]| = |\mathbb{R}|$

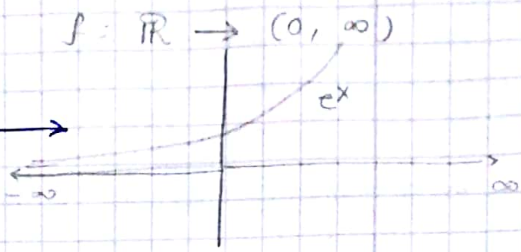
$f: (-\infty,0) \to \mathbb{R}$

$f(x) = \ln(-x)$

is bijective



$f: \mathbb{R} \to (0,\infty)$



$\therefore |(-\infty,0)| = |\mathbb{R}|$

$(-\infty,0) \cup \{0\} = (-\infty,0]$

$|(-\infty,0]| = |(-\infty,0)| = |\mathbb{R}|$

So $|(-\infty,0]| = |\mathbb{R}|$

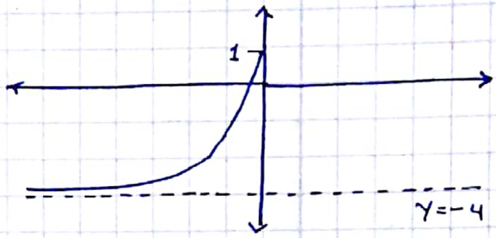
$(0,\infty) \to (-4,1)$

second: show $|(-\infty,0]| = |(-4,1]|$

$g: (-\infty,0] \to (-4,1]$

$g(x) = 5e^x - 4$

is bijective

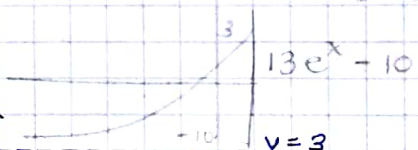


$\therefore |(-\infty,0]| = |(-4,1]|$ and $|(-\infty,0]| = |\mathbb{R}|$

Hence $|\mathbb{R}| = |(-4,1]|$

Since $\mathbb{R}$ is uncountable, $(-4,1]$ is also uncountable

$(-\infty,0) \to (-10,3)$



ii. Convince me that $|(-10,3)| =$ ~~0.00~~ $|(0, 0.0025)|$.

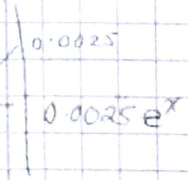
ans: first: show $|(0,\infty)| = |(-10,3)|$

$f: (0,\infty) \to (-10,3)$

$f(x) = -13e^{-x} + 3$

is bijective

$(-\infty,0) \to (0, 0.0025)$



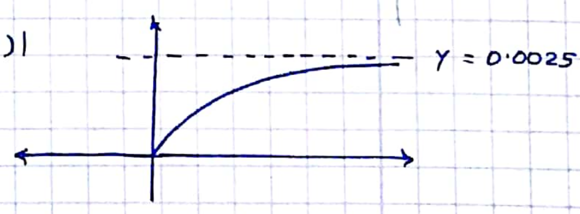
$\therefore |(0,\infty)| = |(-10,3)|$

Second: show $|(0,\infty)| = |(0, 0.0025)|$

$g: (0,\infty) \to (0, 0.0025)$

$g(x) = -0.0025e^{-x} + 0.0025$

is bijective



$\therefore |(0,\infty)| = |(0, 0.0025)|$ and $|(0,\infty)| = |(-10,3)|$

Hence $|(-10,3)| = |(0, 0.0025)|$

iii) Is the set $A = (0, 0.000033) \cap \mathbb{Q}$ finite or infinite. Is A countable?

ans. Betw. any two rational numbers there are infinitely many rational numbers so A is infinite. $A \subset \mathbb{Q}$. Since $\mathbb{Q}$ is countable, so is A.

Rule. subset of countable set $\to$ countable eg $(-\frac{1}{2}, \frac{1}{2}) \subset \mathbb{Q} \to$ countable

subset of uncountable set $\to$ countable or uncountable.

eg. $\mathbb{Q} \subset \mathbb{R} \to$ countable, $\mathbb{Z} \subset \mathbb{R} \to$ countable, $(0,1) \subset \mathbb{R} \to$ uncountable

How to convert betw. bases:

eg: $(124)_{10} \longrightarrow (?)_5$

ans:

$$\begin{array}{r} 24 \\ 5 \overline{)\,124} \\ -10 \\ \hline 24 \\ -20 \\ \hline 4 \end{array} \text{ Stop} \;\longrightarrow\; \begin{array}{r} 4 \\ 5 \overline{)\,24} \\ -20 \\ \hline 4 \end{array} \text{ stop} \;\longrightarrow\; \begin{array}{r} 0 \\ 5 \overline{)\,4} \\ -0 \\ \hline 4 \end{array} \text{ stop}$$

stop.

Read digits right to left (backward) ~~(125~~ $(124)_{10} = (444)_5$

check: ~~10 | 444~~. $4\times5^2 + 4\times5^1 + 4\times5^0$ ⟵ Shorter !

$= 124$

eg $(1236)_7 \longrightarrow (?)_{10}$

$\cancel{1\times10^3}$ $1\times7^3 + 2\times7^2 + 3\times7^1 + 6$

$=(468)_{10}$

eg: $(145)_7 = (?)_{10}$

ans: $1\times7^2 + 4\times7 + 5$

$= (82)_{10}$

go back:

$$\begin{array}{r} 11 \\ 7 \overline{)\,82} \\ -77 \\ \hline 5 \end{array} \;\rightarrow\; \begin{array}{r} 1 \\ 7 \overline{)\,11} \\ -7 \\ \hline 4 \end{array} \;\rightarrow\; \begin{array}{r} 0 \\ 7 \overline{)\,\cancel{4}1} \\ -0 \\ \hline \cancel{4}1 \end{array} \text{ (stop)}$$

$\therefore$ $(82)_{10} = (145)_7$

1-April-2018 base 11: 0, 1, ..., 9, A ↳ 10

base 12: 0, 1, ..., 9, A, B ↘ 11 ~~10~~
↳ 10

⋮

base 16: 0, ..., 9, A, B, C, D, E, F

* use letters to represent numbers above 9 with one character.